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Spectrum, Phase and Group Velocities of the Theoretical Seismograms and the Idea of the Equivalent Surface Source of Disturbance†

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Summary

Analyses are made of theoretical seismograms of torsional oscillation caused by a localized surface source for a homogeneous sphere and a homogeneous mantle with a liquid core. Special care is paid to the effect of parameters on the accuracy of analysis and the limitation of the applicability of various methods. At first, a simple description is given about the assumptions under which the theoretical seismograms were computed ~~(Section 1)~~. Next, the spectrum analysis of the seismograms is carried on by the usual method of Fourier transform, giving useful information about how to employ the method. This analysis corroborates recent similar analyses of the Chilean shock of May 1960 ~~(Section 2)~~. Group and phase velocities calculated by methods ordinarily employed show good agreement with theory. Calculations of initial phase also agree well with theory, giving confidence to similar studies using waves from earthquakes and explosions ~~(Section 3)~~. The idea of an equivalent surface source is proposed relative to the problem of the depth of the source within the Earth ~~(Section 4)~~.

1. Introduction

In a previous study (Satô, Usami & Ewing 1962) disturbances were numerically calculated at various points on the surface of a sphere, assuming a torsional force

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3. Lamont Geological Observatory, Columbia University, Palisades, New York.

on the surface. Since this is a kind of theoretical seismogram, it can be assumed to be relatively free from noise and from uncertainty related to the source mechanism and the dispersion relation. Consequently, by treating this seismogram as if it were an ordinary one observationally obtained, it will be possible to find where errors are introduced by the standard techniques of analysis in common use, to determine the precision of each step, to measure the effect of various parameters on the accuracy of analysis, and to estimate the limits of the applicability of these analytical methods. The results of these analyses are closely related to several recent works relating to the Chilean earthquake, as well as to studies of the data from earthquakes and explosions.

The theoretical seismograms used in this paper were calculated under the following assumptions:

(1) The medium is either a homogeneous elastic sphere, or a homogeneous elastic mantle with a liquid core (Satô, Usami, Landisman & Ewing 1963).

(2) Uniform shear stress is applied on the surface in such a way that

$$\phi r = \begin{cases} 0 & \theta < \theta_1, \quad \theta > \theta_2 \\ 1 & \theta_1 < \theta < \theta_2. \end{cases} \tag{1.1}$$

(3) The time dependence of the source function is

$$f(t) = \begin{cases} 0 & t < t_1, \quad t > t_1 \\ 1/2t_1 & -t_1 < t < t_1. \end{cases} \tag{1.2}$$

(4) Axial symmetry around the pole is assumed, hence $m = 0$ and only the azimuthal component exists.

Calculations are carried out using non-dimensional quantities, but for clarity and convenience, the most frequently used quantities are tabulated in terms of ordinary units:

a (radius of the sphere) = 6370 km

$2\pi a$ (circumference of the sphere) = 40000 km

V_s (velocity of S wave) = 6·667 km/s.

As a result of employing the above values, it is easily seen that

$(2\pi a)/V_s$ (unit of time) = 100 min

Duration of the shear stress being applied = 3 min

Diameter of the area where the stress applied = 444 km.

The following notations are used in this paper:

$_iT_n$: Torsional mode of the free oscillation, i and n being the radial mode number and the order of the oscillation, respectively.

$_iw_n$: Azimuthal displacement due to $_iT_n$ mode

$(_iw_n)$: $\sum_i \sum_n {_iw_n}$

$f(t)$: Time function of the force applied

$f^*(p)$: Fourier conjugate of the function $f(t)$.

2. Spectrum analysis

The theoretical seismogram is the superposition of the various free oscillations having incommensurable discrete frequencies, while the analysis is based on the Fourier transform of a continuous spectrum. However, the spectrum analysis of the theoretical seismogram mentioned above gives a satisfactory result, and the following things can be learned from this analysis.

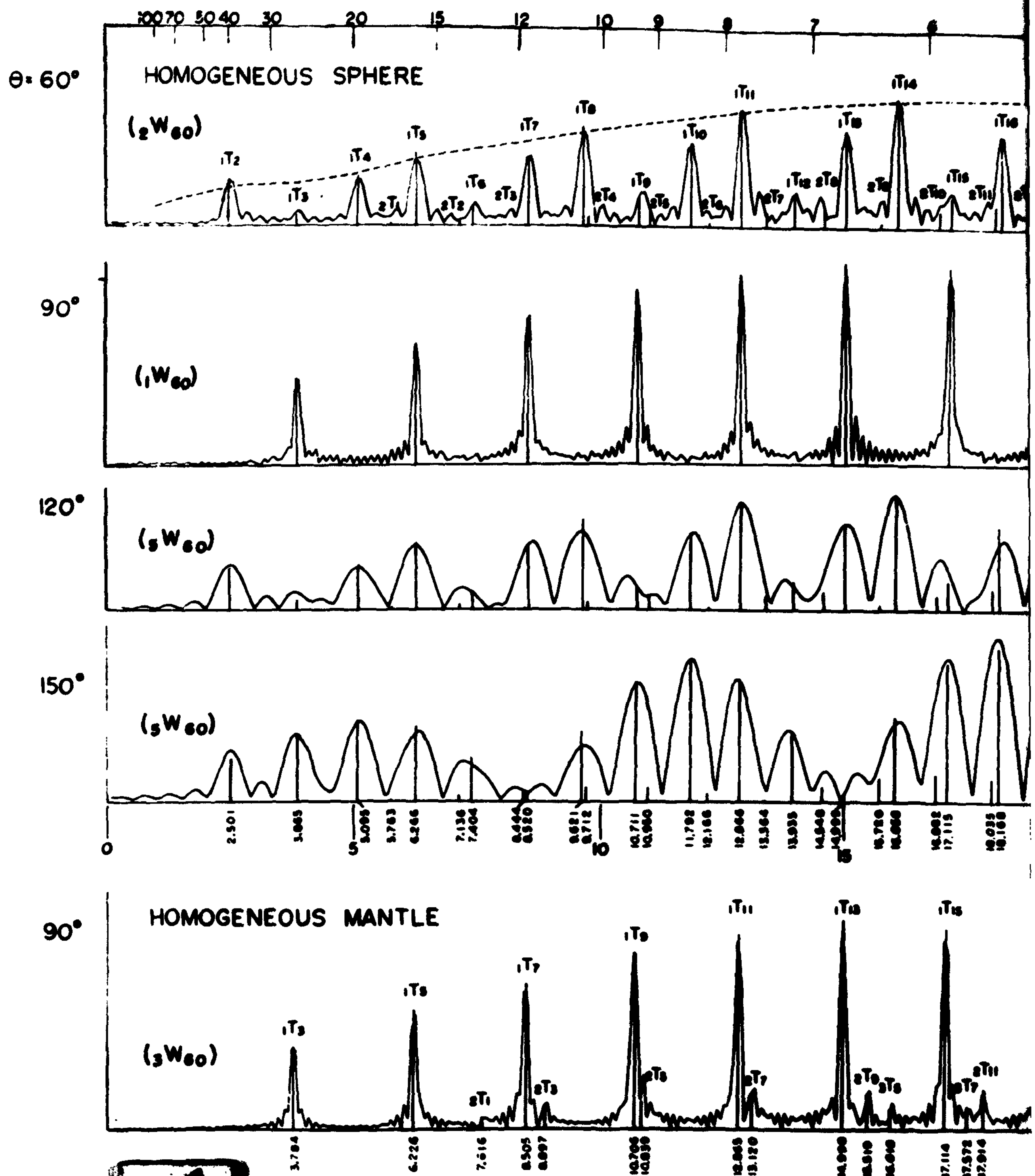


FIG. 1.—Spectrum computed from theoretical seismog[...]
component of displacement due to the torsional oscill[...]
lines, the line spectra of the free torsional oscillation, sh[...]
and the intensity of the spectrum used for computing th[...]
Length of the seismograms used in the analysis is, from[...]
4·0, 7·2, 2·0, 2·0 and 8·0, respectively, in terms of the tim[...]

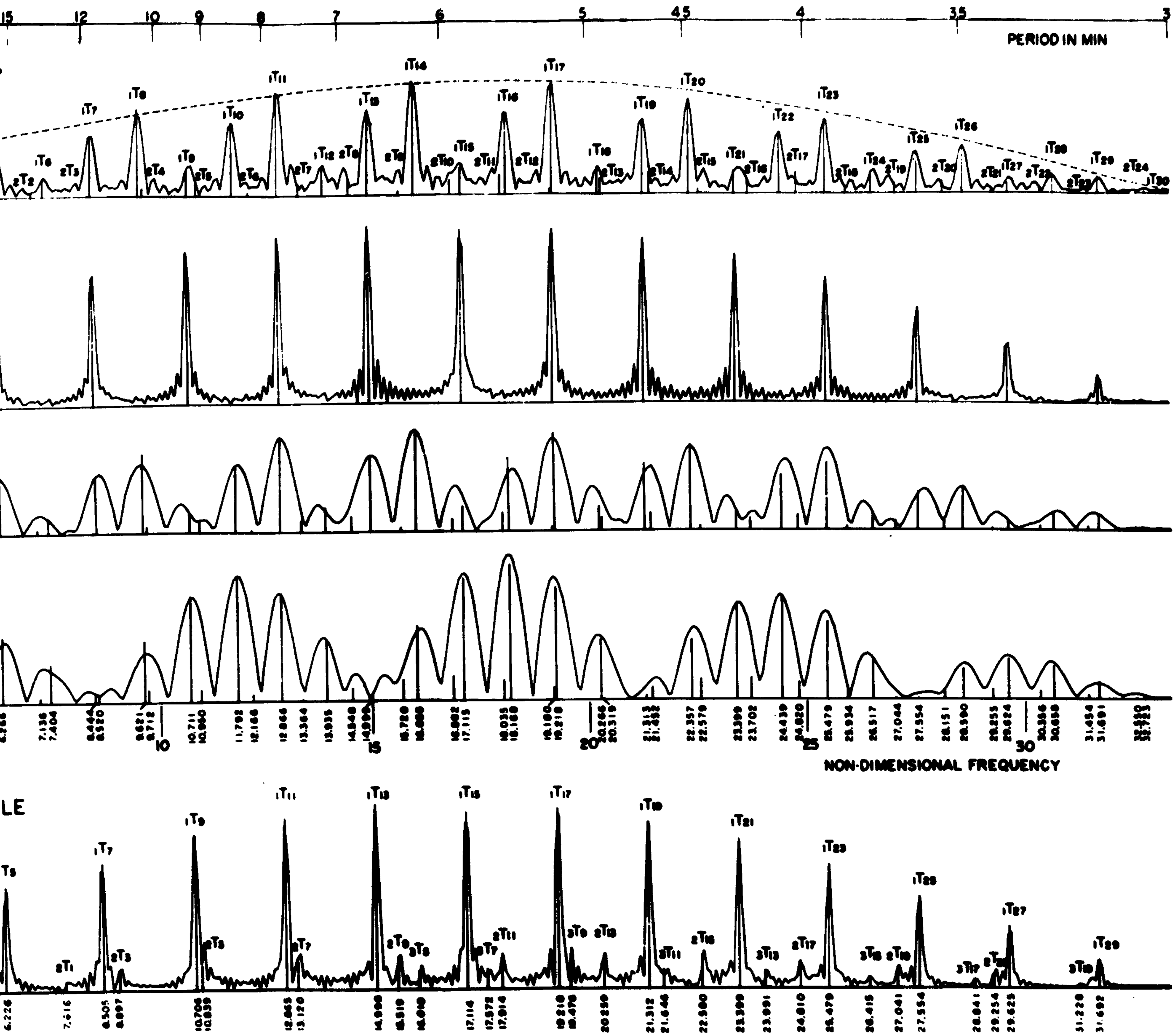


FIG. 1.—Spectrum computed from theoretical seismograms (azimuthal component of displacement due to the torsional oscillation). Vertical lines, the line spectra of the free torsional oscillation, show the frequency and the intensity of the spectrum used for computing the seismograms. Length of the seismograms used in the analysis is, from top to bottom, 4·0, 7·2, 2·0, 2·0 and 8·0, respectively, in terms of the time unit ($2\pi a/V_s$).

[Facing page 2

(1) Both frequency and the amplitudes are given accurately. (See Figure 1. Vertical lines are the line spectra of the free oscillations.)

(2) The resolving power, or the sharpness of the peaks depends on the length of the data analysed, which fact agrees with theory. However, the very long duration of data usually used* seems not to be necessary for the detection of large peaks, unless a very good resolving sharpness is needed.

(3) When there is a large peak, the small peaks nearby are affected, and the frequency and the amplitude are not necessarily given accurately.

(4) The relative heights of the spectral peaks differ from place to place. For instance $\theta = 60°$ and $120°$, which are complementary angles, have similar peaks.

(5) The series of peaks naturally makes a guiding curve (see the fourth curve in Figure 1), which gives a clue for the determination of the mechanism of the origin. This will be the topic of a future study (Satô & Usami 1963).

3. Group and phase velocities

By the ordinary method of measuring the peaks and troughs of the observed disturbance, the apparent group velocity was obtained from the theoretical seismogram (Figure 2). There is no appreciable discrepancy between the curve thus obtained and the theoretically calculated values from the formula

$$U = dz/dn. \tag{3.1}$$

In this expression z is the non-dimensional frequency of free torsional oscillation of a homogeneous elastic sphere, which is tabulated in a previous work of the present authors (Satô & Usami 1962).

Since the phase velocity and the group velocity are connected by a first order differential equation, the phase velocity can be obtained by integrating the curve

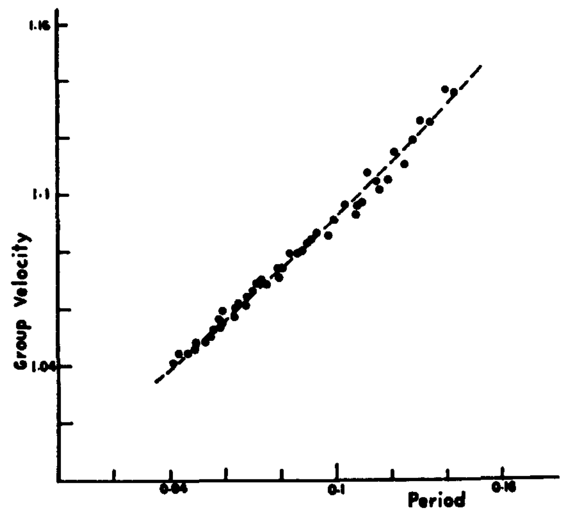


FIG. 2.—Graphs of group velocity (U/V_s) against period for the case of a homogeneous sphere. Solid circles show the group velocity computed by ordinary method of measuring the peaks and troughs of seismograms. Broken line is the curve theoretically obtained from (3.1).

*See for example Benioff & others 1961, Ness & others 1961, Alsop & others 1961, Brune & others 1961.

in Figure 2. The initial value at the starting point, which is unknown, is assumed to be 1·08, 1·09 and 1·10 at (period) = 0·04. All of these curves are shown in Figure 3, from which it is possible to estimate the correct value of the phase velocity.

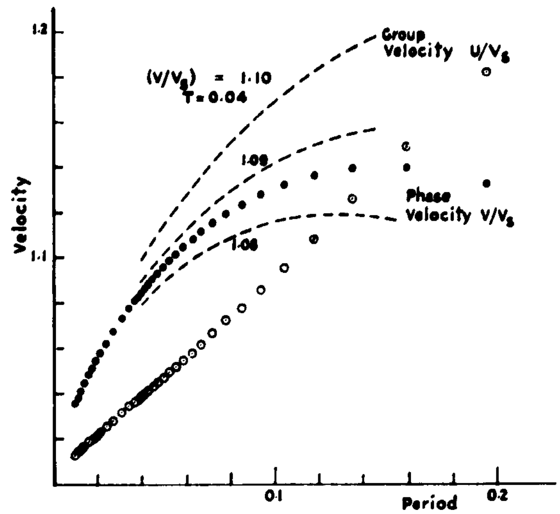


FIG. 3.—Phase velocity (broken lines) obtained by integrating the group velocity curve given in Fig. 2. Open and solid circles are theoretical group and phase velocities for the case of a homogeneous sphere computed from the non-dimensional frequencies of the fundamental mode of the free torsional oscillation. Wavelength is assumed to be equal to $(2\pi a)/(n+\frac{1}{2})$.

3a. *Phase velocity*

The phase velocity can also be calculated using another method (Satô, 1955, 1956, 1960 1960a), namely, the Fourier analysis of two different wave groups, say *G*1 and *G*3. This commonly used process was employed in the present study. Since there are many possible combinations of wave groups, only a number of them are illustrated. Figure 4 contains seismograms of wave groups used in this analysis. The calculated dispersion curve (Figure 5) agrees well with the theoretically obtained one, but discrepancies appear when

(1) The period (wavelength) becomes longer

(2) The difference of two epicentral distances becomes smaller.

The relation between the wavelength and the difference of epicentral distances at which the discrepancy becomes noticeable (about 0·3 per cent) is briefly given as follows (Figure 6 and the following table).

Difference of epicentral distance (Δ)	720°	540°	360°	180°	120°	60°
Wavelength (Λ)	65°	55°	42°	34°	23°	12°
Order of oscillation (*n*)	5	6	8	10	15	18

This relation is summarized by the following empirical formula:

$$\Lambda = 0{\cdot}2\,\Delta \qquad \Delta < 180^\circ \tag{3.2a}$$

$$\Lambda = 23 + 0{\cdot}06\,\Delta \qquad \Delta > 180^\circ \tag{3.2b}$$

If it is assumed that the numerical computation for the phase angle of the Fourier transform of the disturbance involves the error of 0·1 rad, the first formula can be theoretically deduced. Any one of a number of effects can cause the above mentioned error, but probably this is close to the error of a single observation.

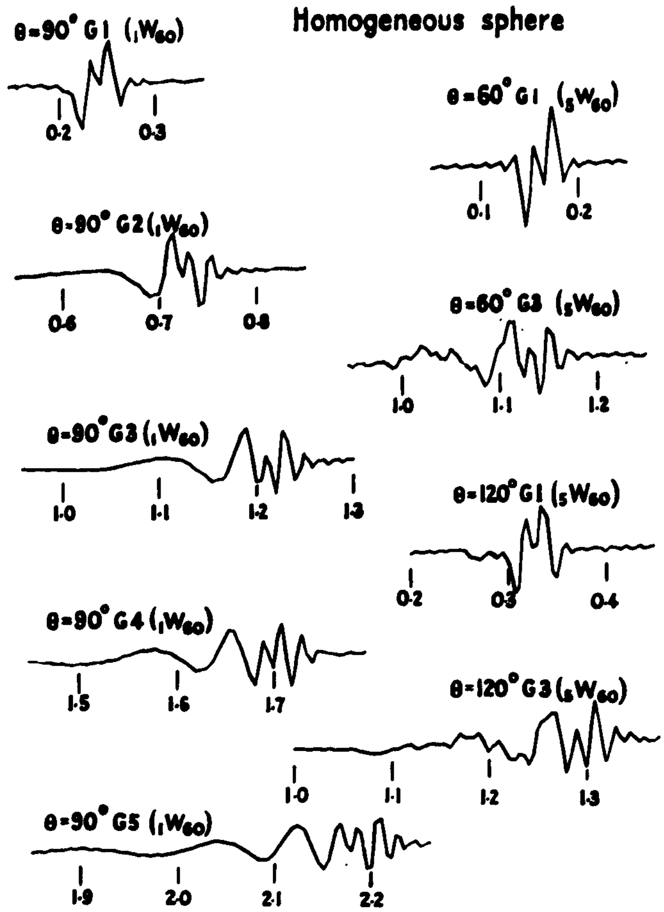


FIG. 4.—Seismograms of wave trains used in the analysis.

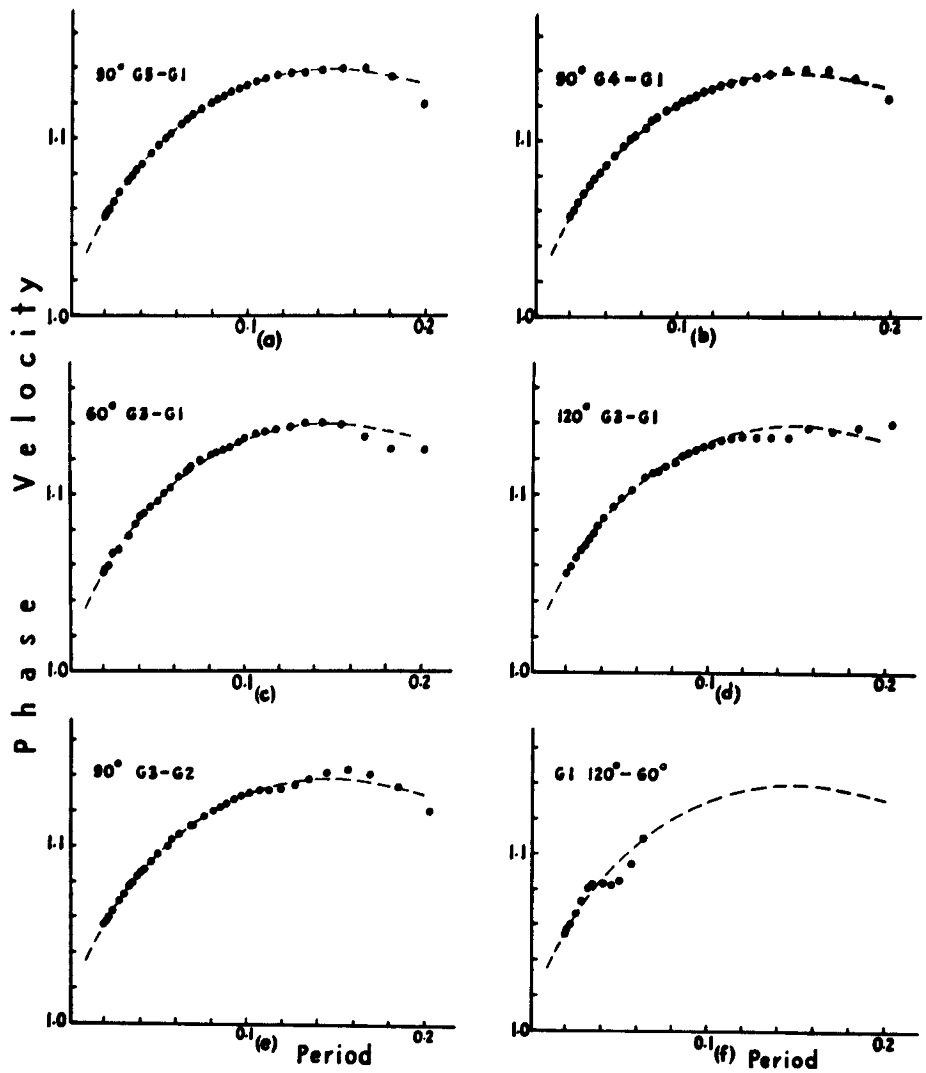
3b. *Spectrum and phase angle of G waves*

Examples of the spectrum and the phase angle of the Fourier transform of *G* waves are shown in Figures 7*a* and 7*b*. Both are restricted to the phase *G*1 only, the range of time analysed being about 0·25, or 25 min. The original disturbance is shown in Figure 4.

From the result of analysis it is seen that:

(1) The spectra at two different points are similar. (Uppermost curve in Figures 7*a* and 7*b*).

(2) There is a zero point in the spectrum curve, which exactly coincides with



FIGS. 5(*a*) to 5(*f*).—Phase velocity (V/V_s) obtained theoretically (broken line) from the frequency of free torsional oscillation and that obtained by Fourier analysis of two different wave groups mentioned in each figure.

the zero point of the Fourier conjugates of the source function, namely

$$(\sin pt_1)/p = 0, \quad t_1 = 0{\cdot}015. \tag{3.3}$$

(3) The distribution of the phase angle is regular, making a smooth and almost straight line. (Middle curve in Figures 7*a* and 7*b*.)

(4) The inclination of this line is proportional to the epicentral distance as predicted by the relation between phase angle, phase velocity and epicentral distance.

(5) There is a jump of π on this line at the zero point of the spectrum.

(6) Since we know the phase velocity, the initial phase $\beta(p)$ at the origin is

calculated from the following formula (Satô, *loc cit*);

$$\beta(p)+L(\theta)+\arg\{f^*(p;\theta)\} = p\theta/V(p) \tag{3.4}$$

in which $L(\theta)$ is the polar phase lag, θ the epicentral distance and $V(p)$ is the phase velocity as a function of frequency p. The curve thus obtained also has a jump of π at the zero point of the spectrum (Figures 7*a* and 7*b* below). On the other hand $\beta(p)$ can be calculated as the argument of the function in (3·3), which should be either zero (for the range of the function $(\sin pt_1)/p$ positive) or π (for the range of the same function negative).

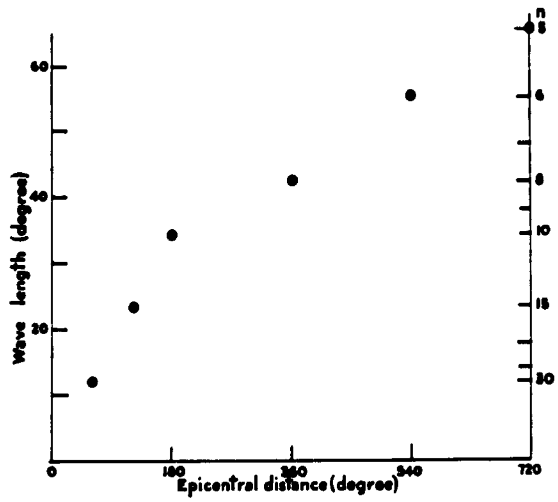


FIG. 6.—Relation between the wavelength and the difference of epicentral distance at which the error of phase velocity calculated by Fourier analysis becomes noticeable (about 0·3 per cent).

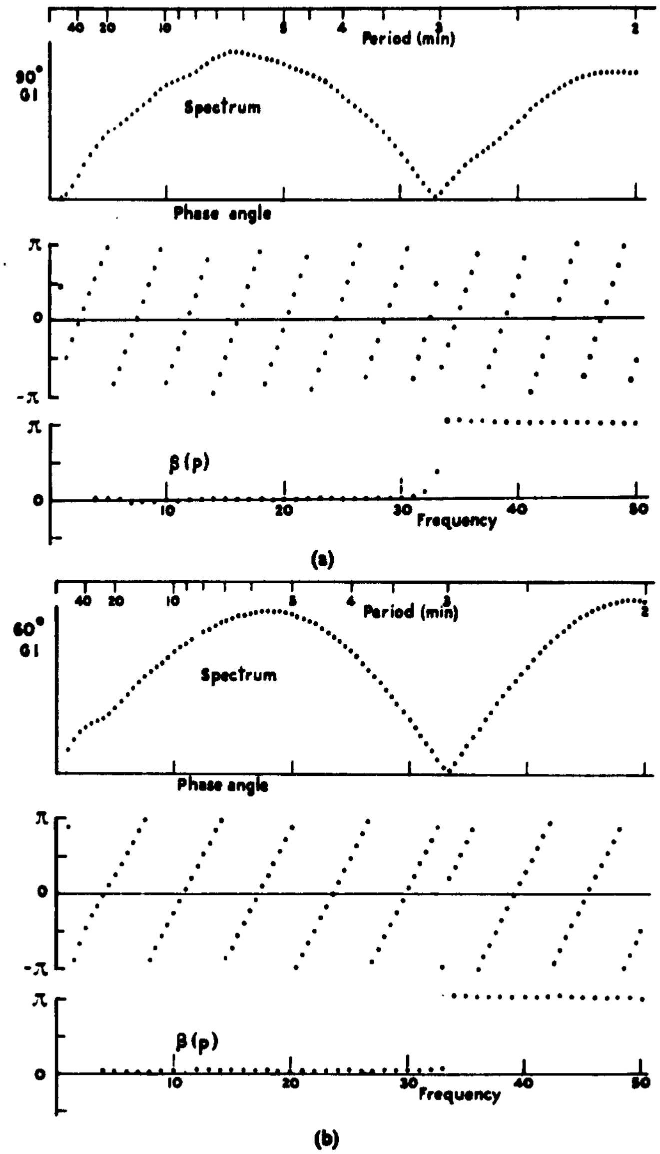
(7) Since there is such a jump of phase angle, care must be taken when the phase velocity is calculated from a single observation.

4. Equivalent surface source

In the previous sections the origin of disturbance was assumed to be on the surface of the sphere. But this is not the case for actual earthquakes and the question naturally arises what will happen if the stress is applied at a surface at some depth within the Earth.

In order to obtain the disturbance when the identical stress is given on the surface $r = b$ instead of $r = a$, the reciprocal theorem of the field is conveniently used. Since it was found in the previous study (Satô, Usami & Ewing 1962) that most of the disturbance belonging to the surface wave is contributed by the fundamental mode, the higher radial modes will be neglected and only the fundamental radial mode will be considered. From the reciprocal theorem, it is known that the displacement at $r = b$ due to the surface source

$$w = \text{constant}\cdot\frac{J_{n+\frac{1}{2}}(kb)}{b^{\frac{1}{2}}}\cdot f^*(p) \tag{4.1}$$



FIGS. 7(*a*) to 7(*b*).—Spectrum, phase angle of the Fourier transform of G waves and β (p) the initial phase at the origin.

is nothing but the surface displacement due to a source at a point $r = b$. In the present case, however, the area on which force is applied is proportional to the square of the radial distance of the source. Therefore, $(b/a)^2$ should be multiplied by (4.1) in order to obtain the surface displacement due to a source at $r = b$,

$$w = \text{constant} \cdot \frac{J_{n+\frac{1}{2}}(kb)}{b^{\frac{1}{2}}} \cdot \left(\frac{b}{a}\right)^2 \cdot f^n(p). \tag{4.2}$$

Multiplying the factor

$$B_n = \left(\frac{b}{a}\right)^{3/2} \cdot \frac{J_{n+\frac{1}{2}}(kb)}{J_{n+\frac{1}{2}}(ka)} \tag{4.3}$$

by the surface displacement due to a surface source (Satô, Usami and Ewing 1962)

$$w_s = \text{constant} \cdot \frac{J_{n+\frac{1}{2}}(ka)}{a^{\frac{1}{2}}} \cdot f^*(p) \tag{4.4}$$

leads to the expression (4.2). This fact implies that the surface displacement due to a force applied at $r = b$ having spectrum $f^*(t)$ is equal to the surface displacement due to a surface force with spectrum $f^*(t) . B_n$. Therefore B_n is termed the "equivalent surface source" which is given in Figure 8.

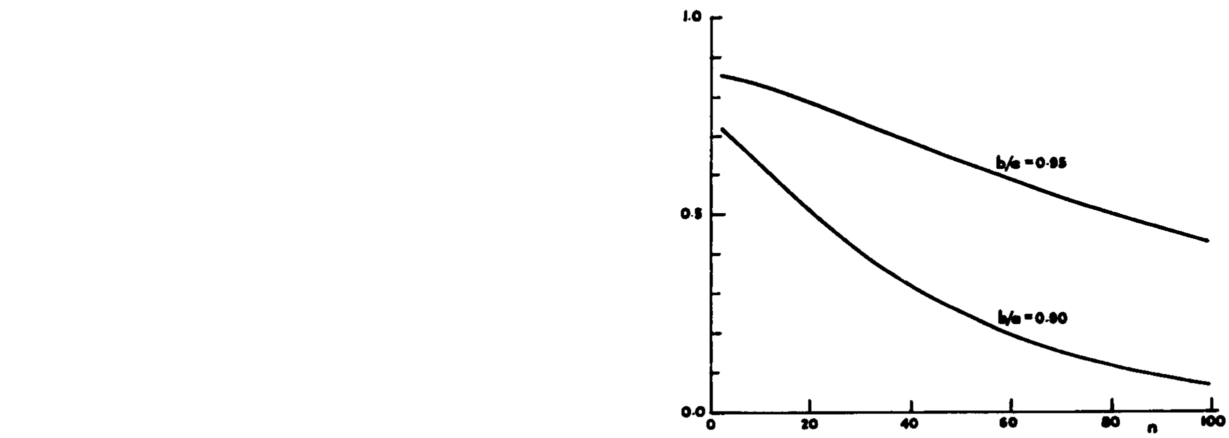


FIG. 8.—Factor B_n. Spectrum of the "equivalent surface source" is obtained by multiplying this function by the spectrum of a source at the depth (a − b).

The time distribution of the stress at the origin was synthesized from the spectrum $(\sin pt_1)/p$ and $B_n . (\sin pt_1)/p$. The first one should give the rectangular form of the original source function. Because of the neglected high frequency (maximum frequency used is 63·66), however, the calculated form of the function is a little different from the original one (Figures 9*a* and 9*b*). When the origin is deep within the Earth, short period surface waves have small amplitude, as is easily expected.

Summary

Theoretical seismograms obtained in previous studies of the propagation of torsional disturbance on the surface of a sphere have been used to test and evaluate the usual methods of analysis of seismograms. Special attention is paid to the accuracy of analysis and the effect of parameters such as length of record and epicentral distance. Both the frequencies and amplitudes of the component spectral lines have been recovered by Fourier analysis. Group and phase velocities obtained by the peak and trough method and by Fourier analysis of successive wave trains show good agreement when compared with theoretical values. The initial phase, important in detection problems, has been recovered, and agrees well with theoretical values. The idea of an equivalent surface source is discussed, using reciprocity to modify the spectrum of a source at depth.

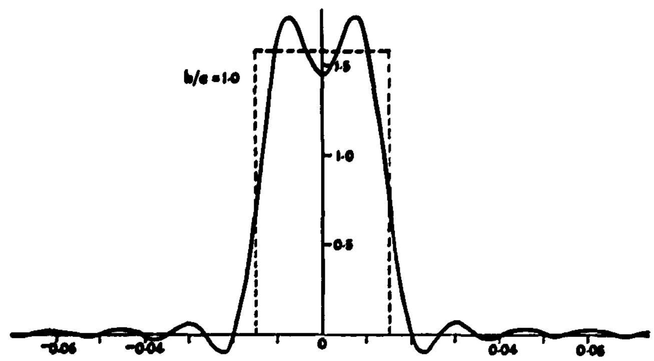


Fig. 9(*a*).—Time distribution of the stress at the origin for the surface source.

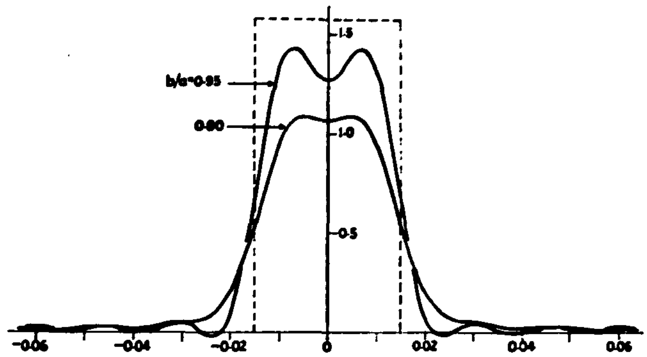


Fig. 9(*b*).—Time distribution of the stress at the origin for the "equivalent surface source," which is given by the expression $B_n . f^*(p)$ (*cf.* (4·3)).

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